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Some Special Functions of Noncommuting Variables

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In this paper we deal with q-commuting variables x and y satisfying the relation $xy = qyx + (q-1)y^2$ with q complex, $0 < |q| < 1$. We study various functional equations for q-exponentials and we deduce some identities for q-special functions involving q-commuting variables.

KEY WORDS: quantum plane; q-binomial formula; q-Heisenberg algebra; basic hypergeometric series.

The q-special functions are extensions to a base q of the standard special functions. In modern treatise of the quantum group as a symmetry group in the quantum space of noncommuting variables (Chari and Pressley, 1994), some q relations of q-special functions are given in the quantum plane generated by two variables x and y satisfying the relation $xy = qyx$, where q is generic. In this paper we take a more general relation of the noncommuting variables x and y satisfying

$$xy = qyx + (q-1)y^2$$

This relation would be considered as a generalization of the q-Heisenberg algebra (Wess, 1999) and it is known that there exists a correspondence between q-Heisenberg algebra and some q-special functions (Biedenharn and Lohe, 1995). From this noncommuting structure and using quantum group arithmetic, we have constructed a new q-special functions. The q-shifted factorials $(a;q)_k$ (the q-extension of the Pochhammer symbol $(a)_k$) are defined by

$$(a;q)_k = \prod_{i=0}^{k-1}(1 - aq^i) \tag{1}$$

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and

$$(a;q)_\infty = \prod_{i=0}^{\infty}(1-aq^i) \tag{2}$$

The q-binomial coefficient is defined by

$$\binom{n}{k}_q = \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}} \quad k = 0, 1, \ldots, n \tag{3}$$

where n is a nonnegative integer. In general, for $\alpha, \beta \in C$, we get

$$\binom{\alpha}{\beta}_q = \frac{\Gamma_q(\alpha+1)}{\Gamma_q(\beta+1)\Gamma_q(\alpha-\beta+1)}$$

where

$$\Gamma_q(z) = \frac{(q;q)_\infty}{(q^z;q)_\infty}(1-q)^{1-z}$$

and

$$\lim_{q\to 1}\Gamma_q(z) = \Gamma(z)$$

The basic hypergeometric series is defined by (Koekoek and Swarttouw, 1994; Koelink, 1996)

$${}_r\Phi_s\left(\begin{matrix} a_1, \ldots, a_r \\ b_1, \ldots, b_s \end{matrix} \mid q;z\right) = \sum_{k=0}^{\infty} \frac{(a_1, \ldots, a_r;q)_k}{(b_1, \ldots, b_s;q)_k}\left((-1)^k q^{\frac{k}{2}(k-1)}\right)^{1+s-r} \frac{z^k}{(q;q)_k} \tag{4}$$

where

$$(a_1, \ldots, a_r;q)_k = \prod_{i=1}^{r}(a_i;q)_k \tag{5}$$

The basic hypergeometric series ${}_r\Phi_s$ is a polynomial in z if one of a_i equals q^{-n}, where n is a nonnegative integer. Otherwise the radius of convergence of series (4) is

$$\rho = \begin{cases} \infty & \text{if } r < s+1 \\ 1 & \text{if } r = s+1 \\ 0 & \text{if } r > s+1 \end{cases}$$

The classical exponential function e^z can be expressed in terms of the hypergeometric functions as $e^z = {}_0F_0(\overline{-};z)$; this function has two different natural q-extensions denoted by $e_q(z)$ and $E_q(z)$ and defined by

$$e_q(z) = {}_1\Phi_0\left(\begin{matrix} 0 \\ - \end{matrix} \mid q;z\right) = \sum_{k=0}^{\infty}\frac{z^k}{(q;q)_k} = \frac{1}{(z;q)_\infty} \tag{6}$$

and

$$E_q(z) = {}_0\Phi_0\left(\begin{matrix}0\\-\end{matrix}\ |; qz\right) = \sum_{k=0}^{\infty}\frac{q^{\binom{k}{2}}z^k}{(q;q)_k} = (-z;q)_\infty \tag{7}$$

where $z \in C$, $|z| < 1$, and $0 < q < 1$. Also $e_q(z)$ and $E_q(z)$ can be considered as formal power series in the formal variable z. They have the following properties (Ahmed *et al.*, 2000):

$e_q(z)E_q(-z) = 1$
$e_q(qz) = (1 - z)e_q(z)$
$E_q(z) = (1 + z)E_q(qz)$
$e_q(z) = (1 - q^{-1}z)e_q(q^{-1}z)$
$E_q(q^{-1}z) = (1 + q^{-1}z)E_q(z)$
$\lim_{q\to 1} e_q((1 - q)z) = \lim_{q\to 1} E_q((1 - q)z) = e^z$

It is known that the Newton binomial formula is given by

$$(x + y)^n = \sum_{k=0}^{n}\binom{n}{k}y^k x^{n-k}$$

for any arbitrary commutative ring generated by 1, x, and y; and the binomial coefficients $\binom{n}{k}$ are integers.

For q-commuting variables x and y such that

$$xy = qyx \tag{8}$$

and some $q \in C$, the q-binomial formula (Koornwinder, 1992, 1994, 1997) is given by

$$(x + y)^n = \sum_{k=0}^{n}\binom{n}{k}_q y^{n-k}x^k \tag{9}$$

Recently, Benaoum (1999) gave the q-binomial formula

$$(x + y)^n = \sum_{k=0}^{n}\binom{n}{k}_q \prod_{j=0}^{k-1}\left(1 + h\sum_{i=0}^{j-1}q^i\right)y^k x^{n-k} \tag{10}$$

for x and y satisfying

$$xy = qyx + hy^2 \tag{11}$$

From the point of view of noncommutative geometry the h-deformed quantum plane ($xy = qyx + hy^2$) is in some ways better than the q-deformed quantum plane ($xy = qyx$), which gives the standard deformation of GL (2) (Celik, 1997).

In the algebra generated by x and y with $xy = qyx + hy^2$ if we consider the case $h = q - 1$, then we get

$$\prod_{j=0}^{k-1}(1 + h\sum_{i=0}^{j-1} q^i) = q^{\binom{k}{2}}$$

Let $C_q[x, y]$ be the complex associative algebra with one of the formal power series $\sum_{i,j=0}^{\infty} c_{i,j} y^j x^i$, with arbitrary complex coefficient $c_{i,j}$, and x, y satisfying

$$xy = qyx + (q-1)y^2 \tag{12}$$

Then q-binomial formula is

$$(x + y)^n = \sum_{k=0}^{n} \binom{n}{k}_q q^{\binom{k}{2}} y^k x^{n-k} \tag{13}$$

The algebra $C_q[x, y]$ is a generalization of the q-Heisenberg algebra. Puting $c = -y^2$ and $q^2 = 1$, one gets the q-Heisenberg algebra

$$xy - qyx = (1-q)c, \quad xc = cx, \quad \text{and} \quad yc = cy$$

The following proposition is a generalization of the classical functional equation $e^x e^y = e^{x+y}$ of commuting variables x and y.

Proposition 1. *In the algebra $C_q[x, y]$*

$$\begin{aligned} (i) \quad & e_q(x + y) = E_q(y)e_q(x) \\ (ii) \quad & E_q(x + y) = E_q(x)e_q(y) \end{aligned} \tag{14}$$

Proof:

$$\begin{aligned} e_q(x + y) &= \sum_{n=0}^{\infty} \frac{(x + y)^n}{(q;q)_n} = \sum_{n=0}^{\infty} \left(\frac{1}{(q;q)_n} \sum_{k=0}^{n} \binom{n}{k}_q q^{\binom{k}{2}} y^k x^{n-k} \right) \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \left(\frac{1}{(q;q)_n} \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}} q^{\binom{k}{2}} y^k x^{n-k} \right) \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \left(\frac{q^{\binom{k}{2}}}{(q;q)_k (q;q)_{n-k}} y^k x^{n-k} \right) \end{aligned}$$

Put $n - k = l$, then

$$\begin{aligned} e_q(x + y) &= \sum_{l,k=0}^{\infty} \left(\frac{q^{\binom{k}{2}}}{(q;q)_k (q;q)_l} y^k x^l \right) \\ &= E_q(y)e_q(x) \end{aligned}$$

To prove (ii) use the relation $e_q(z)E_q(-z) = 1$. Then

$$E_q(x + y)e_q(-x - y) = 1$$

and

$$\begin{aligned} E_q(x + y) &= e_q^{-1}(-x - y) = (E_q(-y)e_q(-x))^{-1} \\ &= e_q^{-1}(-x)E_q^{-1}(-y) = E_q(x)e_q(y) \end{aligned} \qquad \square$$

Proposition 2. *In the algebra* $C_q[x, y]$

$$e_q(x)e_q(y) = e_q(y)e_q((1 - y)x - y^2) \tag{15}$$

Proof: For any two variables x and y (noncommuting or commuting) if $f(x)$ and $g(y)$ are formal power series with $f(x)$ invertible and $g(y) = \sum_{l,k=0}^{\infty} c_n y^n$, then by induction

$$\begin{aligned} &\text{(i)} \quad (f(x)yf^{-1}(x))^n = f(x)y^n f^{-1}(x) \\ &\text{(ii)} \quad f(x)g(y)f^{-1}(x) = g(f(x)yf^{-1}(x)) \end{aligned} \tag{16}$$

Now in the algebra $C_q[x, y]$

$$\begin{aligned} xy^n &= (qyx + hy^2)y^{n-1} \\ &= qyxy^{n-1} + hy^{n+1} \\ &= qy(qyx + hy^2)y^{n-2} + hy^{n+1} \\ &= q^2y^2xy^{n-2} + (q + 1)hy^{n+1} \\ &= q^3y^3xy^{n-3} + (q^2 + q + 1)hy^{n+1} \\ &= \cdots = q^n y^n x + \left(\sum_{j=0}^{n-1} q^j\right) hy^{n+1} \end{aligned}$$

Put $h = q - 1$, then

$$\left(\sum_{j=0}^{n-1} q^j\right) h = q^n - 1$$

Then

$$xy^n = q^n y^n x + (q^n - 1)y^{n+1} \tag{17}$$

Now by using (17)

$$
\begin{aligned}
xe_q(y) &= \sum_{n=0}^{\infty} \frac{xy^n}{(q;q)_n} \\
&= \sum_{n=0}^{\infty} \frac{1}{(q;q)_n}(q^n y^n x + (q^n - 1)y^{n+1}) \\
&= \sum_{n=0}^{\infty} \frac{(qy)^n}{(q;q)_n} x + y \sum_{n=0}^{\infty} \frac{(qy)^n}{(q;q)_n} - y \sum_{n=0}^{\infty} \frac{y^n}{(q;q)_n} \\
&= e_q(qy)x + ye_q(qy) - ye_q(y) \\
&= e_q(y)((1-y)x - y^2)
\end{aligned}
$$

then

$$
\begin{aligned}
e_q^{-1}(y)xe_q(y) &= (1-y)x - y^2 \\
e_q(e_q^{-1}(y)xe_q(y)) &= e_q((1-y)x - y^2)
\end{aligned}
$$

By using (16) we get

$$
\begin{aligned}
e_q^{-1}(y)(y) &= e_q((1-y)x - y^2) \\
e_q(x)e_q(y) &= e_q(y)e_q((1-y)x - y^2)
\end{aligned}
$$

□

By the same way we can prove the following proposition.

Proposition 3. *In the algebra* $C_q[x, y]$

$$E_q(x)e_q(y) = e_q(y)E_q((1-y)x - y^2) \tag{18}$$

$$e_q(x)E_q(y) = E_q(y)e_q\left(\frac{x - y^2}{1 + y}\right) \tag{19}$$

$$E_q(x)E_q(y) = E_q(y)E_q\left(\frac{x - y^2}{1 + y}\right) \tag{20}$$

$$e_q(y)e_q(x) = e_q\left(\frac{x + q^{-1}y^2}{1 - q^{-1}y}\right)e_q(y) \tag{21}$$

$$e_q(y)E_q(x) = E_q\left(\frac{x + q^{-1}y^2}{1 - q^{-1}y}\right)e_q(y) \tag{22}$$

$$E_q(y)E_q(x) = E_q(x(1 + q^{-1}y) + q^{-1}y^2)E_q(y) \tag{23}$$

$$E_q(y)e_q(x) = e_q(x(1 + q^{-1}y) + q^{-1}y^2)E_q(y) \tag{24}$$

The nonterminating q-binomial series is

$$\begin{aligned} {}_1\Phi_0(\underline{a} \quad | \, q; z) &= \sum_{k=0}^{\infty} \frac{z^k (a; q)_k}{(q; q)_k} = \frac{(az; q)_\infty}{(z; q)_\infty} \\ &= E_q(-az) e_q(z) = e_q(z) E_q(-az) \end{aligned} \tag{25}$$

The functional equation $(1-x)^{-a}(1-y)^{-a} = (1-x-y+xy)^{-a}$ in the algebra of commuting variables x and y can be written in the form

$${}_1F_0(\underline{a} \quad | \, x) {}_1F_0(\underline{a} \quad | \, y) = {}_1F_0(\underline{a} \quad | \, x+y-xy)$$

we now give two q-analogues of this equation, valid in the algebra $C_q[x, y]$.

Proposition 4. *In the algebra $C_q[x, y]$*

$${}_1\Phi_0(\underline{a} \quad | \, q; x) {}_1\Phi_0(\underline{a} \quad | \, q; y) = e_q(y) {}_1\Phi_0(\underline{a} \quad | \, q; (1-y)x - y^2) E_q(-ay) \tag{26}$$

$${}_1\Phi_0(\underline{a} \quad | \, q; y) {}_1\Phi_0(\underline{a} \quad | \, q; x) = E_q(-ay) {}_1\Phi_0 \left(\underline{a} \quad | \, q; \frac{x + q^{-1} y^2}{1 - q^{-1} y^2} \right) e_q(-ay) \tag{27}$$

Proof: (i)

$$\begin{aligned} e_q^{-1}(y) x e_q(y) &= (1-y)x - y^2 \\ e_q^{-1}(y)(-ax) e_q(y) &= -a[(1-y)x - y^2] \end{aligned}$$

then

$$\begin{aligned} {}_1\Phi_0(\underline{a} \quad | \, q; x) {}_1\Phi_0(\underline{a} \quad | \, q; y) &= e_q(x) E_q(-ax) e_q(y) E_q(-ay) \\ &= E_q(-ax) e_q(x) e_q(y) E_q(-ay) \\ &= E_q(-ax) e_q(y) e_q((1-y)x - y^2) E_q(-ay) \\ &= e_q(y) E_q(-a[(1-y)x - y^2]) \\ &\quad \times e_q((1-y)x - y^2) E_q(-ay) \\ &= e_q(y) {}_1\Phi_0(\underline{a} \quad | \, q; (1-y)x - y^2) E_q(-ay) \end{aligned}$$

(ii)

$$\begin{aligned} e_q(y) x e_q^{-1}(y) &= \frac{x + q^{-1} y^2}{1 - q^{-1} y} \\ e_q(y)(-ax) e_q^{-1}(y) &= -a \left[\frac{x + q^{-1} y^2}{1 - q^{-1} y} \right] \end{aligned}$$

then

$$e_q(y)E_q(-ax)e_q^{-1}(y) = E_q\left(-a\left[\frac{x+q^{-1}y^2}{1-q^{-1}y}\right]\right)$$

$$e_q(y)E_q(-ax) = E_q\left(-a\left[\frac{x+q^{-1}y^2}{1-q^{-1}y}\right]\right)e_q(y)$$

$$\begin{aligned}
{}_1\Phi_0(\underline{a} \mid q;y)\,{}_1\Phi_0(\underline{a} \mid q;x) &= E_q(-ay)e_q(y)e_q(x)E_q(-ax) \\
&= E_q(-ay)e_q\left(\frac{x+q^{-1}y^2}{1-q^{-1}y}\right)e_q(y)E_q(-ax) \\
&= E_q(-ay)e_q\left(\frac{x+q^{-1}y^2}{1-q^{-1}y}\right) \\
&\quad \times E_q\left(-a\left[\frac{x+q^{-1}y^2}{1-q^{-1}y}\right]\right)e_q(y) \\
&= E_q(-ay)\,{}_1\Phi_0\left(\underline{a} \mid q;\frac{x+q^{-1}y^2}{1-q^{-1}y}\right)e_q(-ay)
\end{aligned}$$

□

Proposition 5. *In the algebra $C_q[x,y]$*

$$\begin{aligned}
&\left(1+\frac{x-y^2}{1+y}\right){}_1\Phi_0\left(\underline{-q} \mid q;\frac{x-y^2}{1+y}\right) \\
&\quad = \left(1+\frac{x-q^{-1}y^2}{1+q^{-1}y}\right){}_1\Phi_0\left(\underline{-q} \mid q;\frac{x-q^{-1}y^2}{1+q^{-1}y}\right)
\end{aligned}$$

Proof:

$$\begin{aligned}
&\left(1+\frac{x-y^2}{1+y}\right){}_1\Phi_0\left(\underline{-q} \mid q;\frac{x-y^2}{1+y}\right) \\
&\quad = \left(1+\frac{x-y^2}{1+y}\right)E_q\left(q\frac{x-y^2}{1+y}\right)e_q\left(\frac{x-y^2}{1+y}\right) \\
&\quad = E_q\left(\frac{x-y^2}{1+y}\right)e_q\left(\frac{x-y^2}{1+y}\right) \\
&\quad = E_q^{-1}(y)E_q(x)E_q(y)E_q^{-1}(y)e_q(x)E_q(y) \\
&\quad = E_q^{-1}(y)E_q(x)e_q(x)E_q(y)
\end{aligned}$$

$$
\begin{aligned}
&= E_q^{-1}(y)e_q(x)E_q(x)E_q(y) \\
&= e_q(-y)e_q(x)E_q(x)E_q(y) \\
&= e_q\left(\frac{x-q^{-1}y^2}{1+q^{-1}y}\right)e_q(-y)E_q(x)E_q(y) \\
&= e_q\left(\frac{x-q^{-1}y^2}{1+q^{-1}y}\right)E_q\left(\frac{x-q^{-1}y^2}{1+q^{-1}y}\right)e_q(-y)E_q(y) \\
&= e_q\left(\frac{x-q^{-1}y^2}{1+q^{-1}y}\right)E_q\left(\frac{x-q^{-1}y^2}{1+q^{-1}y}\right) \\
&= E_q\left(\frac{x-q^{-1}y^2}{1+q^{-1}y}\right)e_q\left(\frac{x-q^{-1}y^2}{1+q^{-1}y}\right) \\
&= \left(1+\frac{x-q^{-1}y^2}{1+q^{-1}y}\right)E_q\left(q\frac{x-q^{-1}y^2}{1+q^{-1}y}\right)e_q\left(\frac{x-q^{-1}y^2}{1+q^{-1}y}\right) \\
&= \left(1+\frac{x-q^{-1}y^2}{1+q^{-1}y}\right){}_1\Phi_0\left(\begin{matrix}-q\\ -\end{matrix}\;\Big|\; q;\frac{x-q^{-1}y^2}{1+q^{-1}y}\right) \qquad \square
\end{aligned}
$$

The big q-Jacobi polynomials (Noumi and Mimachi, 1990a,b) are defined by

$$
P(x;a,b,c;q) = {}_3\Phi_2\left(\begin{matrix}q^{-n}, abq^{n+1}, x\\ aq, cq\end{matrix}\;\Big|\; q;q\right) \tag{28}
$$

Now we will give the following identity for the big q-Jacobi polynomials in q-commuting variables and we will generalize it to the general form of ${}_r\Phi_s$.

Proposition 6. *In the algebra $C_q[x,y]$*

$$
P(x+y;a,b,c;q) = E_q(y){}_4\Phi_3\left(\begin{matrix}q^{-n}, abq^{n+1}, x, 0\\ aq, cq, -y\end{matrix}\;\Big|\; q;q\right)e_q(-y)
$$

Proof: At first

$$
\begin{aligned}
E_q(q^n z) = (-q^n z;q)_\infty &= \prod_{i=0}^{\infty}(1-q^i q^n(-z)) \\
&= \prod_{i=0}^{\infty}(1-q^{i+n}(-z)) = \prod_{i=n}^{\infty}(1-q^i(-z)) \\
&= \frac{\prod_{i=0}^{\infty}(1-q^i(-z))}{\prod_{i=0}^{n-1}(1-q^i(-z))} = \frac{E_q(z)}{(-z;q)_n}
\end{aligned}
$$

Then

$$E_q(q^n z) = \frac{E_q(z)}{(-z;q)_n}$$

By the same way

$$e_q(q^n z) = (z;q)_n e_q(z)$$

Also

$$(z;q)_n = e_q(q^n z) E_q(-z)$$

Now we will get $(x+y;q)_n$

$$\begin{aligned}
(x+y;q)_n &= e_q(q^n[x+y]) E_q(-x-y) \\
&= E_q(q^n y) e_q(q^n x) E_q(-x) e_q(-y) \\
&= \frac{E_q(y)}{(-y;q)_n} (x;q)_n e_q(x) E_q(-x) e_q(-y) \\
&= E_q(y) \frac{(x;q)_n}{(-y;q)_n} e_q(-y)
\end{aligned}$$

Now we will get $P(x+y;a,b,c;q)$.

$$\begin{aligned}
P(x+y;a,b,c;q) &= {}_3\Phi_2 \left(\begin{matrix} q^{-n}, abq^{n+1}, x+y \\ aq, cq \end{matrix} \mid q;q \right) \\
&= \sum_{k=0}^{\infty} \frac{(q^{-n}, abq^{n+1}, x+y;q)_k}{(aq,cq;q)_k} \frac{q^k}{(q;q)_k} \\
&= \sum_{k=0}^{\infty} \frac{(q^{-n};q)_k (abq^{n+1};q)_k (x+y;q)_k}{(aq,cq;q)_k} \frac{q^k}{(q;q)_k} \\
&= \sum_{k=0}^{\infty} \frac{(q^{-n};q)_k (abq^{n+1};q)_k}{(aq,cq;q)_k} E_q(y) \frac{(x;q)_n}{(-y;q)_n} \\
&\quad \times e_q(-y) \frac{q^k}{(q;q)_k} \\
&= E_q(y) \left[\sum_{k=0}^{\infty} \frac{(q^{-n};q)_k (abq^{n+1};q)_k (x;q)_n}{(aq,cq;q)_k (-y;q)_n} \frac{q^k}{(q;q)_k} \right] e_q(-y) \\
&= E_q(y) {}_4\Phi_3 \left(\begin{matrix} q^{-n}, abq^{n+1}, x, 0 \\ aq, cq, -y \end{matrix} \mid q;q \right) e_q(-y) \qquad \square
\end{aligned}$$

The above result can be generalized as

Lemma 7. *In the algebra $C_q[x, y]$*

$$
{}_r\Phi_s\left(\begin{matrix} a_1, \ldots, a_{r-1}, x+y \\ b_1, \ldots, b_s \end{matrix} \mid q; z\right) = E_q(y)\,{}_{r+1}\Phi_{s+1}\left(\begin{matrix} a_1, \ldots, a_{r-1}, x, 0 \\ b_1, \ldots, b_s, -y \end{matrix} \mid q; z\right) \times e_q(-y) \tag{29}
$$

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